Stochastic CoSaMP: Randomizing Greedy Pursuit for Sparse Signal Recovery: Appendix

Anonymous Author(s)
Affiliation
Address
email

Abstract

Here we provide detailed proofs of results in the paper.

1 Proof of Lemma 3

Proof. We have $F = \Omega \cup \Psi$. From the block inversion formula,

$$(\mathbf{\Phi}_F^* \mathbf{\Phi}_F)^{-1} = \begin{pmatrix} \mathbf{\Phi}_\Omega^* \mathbf{\Phi}_\Omega & \mathbf{\Phi}_\Omega^* \mathbf{\Phi}_\Psi \\ \mathbf{\Phi}_\Psi^* \mathbf{\Phi}_\Omega & \mathbf{\Phi}_\Psi^* \mathbf{\Phi}_\Psi \end{pmatrix}^{-1} \tag{1}$$

$$= \begin{pmatrix} \mathbf{P} & \mathbf{Q} \\ \mathbf{R} & \mathbf{S} \end{pmatrix} \tag{2}$$

, where

$$\mathbf{P} = (\mathbf{\Phi}_\Omega^* \mathbf{\Phi}_\Omega)^{-1} + (\mathbf{\Phi}_\Omega^* \mathbf{\Phi}_\Omega)^{-1} \mathbf{\Phi}_\Omega^* \mathbf{\Phi}_\Psi \mathbf{\Pi}_\Omega^{-1} \mathbf{\Phi}_\Psi^* \mathbf{\Phi}_\Omega (\mathbf{\Phi}_\Omega^* \mathbf{\Phi}_\Omega)^{-1} \tag{3}$$

$$\mathbf{Q} = -(\mathbf{\Phi}_\Omega^* \mathbf{\Phi}_\Omega)^{-1} \mathbf{\Phi}_\Omega^* \mathbf{\Phi}_\Psi \mathbf{\Pi}_\Omega^{-1} \tag{4}$$

$$\mathbf{R} = -\mathbf{\Pi}_\Omega^{-1} \mathbf{\Phi}_\Psi^* \mathbf{\Phi}_\Omega (\mathbf{\Phi}_\Omega^* \mathbf{\Phi}_\Omega)^{-1} \tag{5}$$

$$\mathbf{S} = \mathbf{\Pi}_\Omega^{-1} \tag{6}$$

$$\mathbf{\Pi}_\Omega = \mathbf{\Phi}_\Psi^* \mathbf{\Phi}_\Psi - \mathbf{\Phi}_\Psi^* \mathbf{\Phi}_\Omega (\mathbf{\Phi}_\Omega^* \mathbf{\Phi}_\Omega)^{-1} \mathbf{\Phi}_\Omega^* \mathbf{\Phi}_\Psi \tag{7}$$

The row corresponding to Ω (i.e.$((\mathbf{\Phi}_F^* \mathbf{\Phi}_F)^{-1})^\Omega$) is given by $\begin{pmatrix} \mathbf{P}\mathbf{x}_{|\Omega} & \mathbf{Q}\mathbf{x}_{|\Psi} \end{pmatrix}$. Our strategy is to bound it from both sides, i.e. find bounds for $||\mathbf{P}\mathbf{x}_{|\Omega}||_2$ and $||\mathbf{Q}\mathbf{x}_{|\Psi}||_2$ and then use

$$||\mathbf{P}\mathbf{x}_{|\Omega}||_2 - ||\mathbf{Q}\mathbf{x}_{|\Psi}||_2 \leq ||((\mathbf{\Phi}_\mathbf{F}^* \mathbf{\Phi}_\mathbf{F})^{-\mathbf{1}})^\Omega \mathbf{x}||_2 \leq ||\mathbf{P}\mathbf{x}_{|\Omega}||_2 + ||\mathbf{Q}\mathbf{x}_{|\Psi}||_2 \tag{8}$$

We have from Equation. 7 and using Woodbury's matrix inversion formula and assuming $\mathbf{\Phi}_\Psi^* \mathbf{\Phi}_\Psi$ is full rank,

$$\mathbf{\Pi}_\Omega^{-1} = (\mathbf{\Phi}_\Psi^* \mathbf{\Phi}_\Psi + \mathbf{\Phi}_\Psi^* \mathbf{\Phi}_\Omega (-\mathbf{\Phi}_\Omega^* \mathbf{\Phi}_\Omega)^{-1} \mathbf{\Phi}_\Omega^* \mathbf{\Phi}_\Psi)^{-1} \tag{9}$$

$$= (\mathbf{\Phi}_\Psi^* \mathbf{\Phi}_\Psi)^{-1} - (\mathbf{\Phi}_\Psi^* \mathbf{\Phi}_\Psi)^{-1} \mathbf{\Phi}_\Psi^* \mathbf{\Phi}_\Omega \mathbf{\Gamma}^{-1} \mathbf{\Phi}_\Omega^* \mathbf{\Phi}_\Psi (\mathbf{\Phi}_\Psi^* \mathbf{\Phi}_\Psi)^{-1} \tag{10}$$

where $\mathbf{\Gamma} = (-(\mathbf{\Phi}_\Omega^* \mathbf{\Phi}_\Omega)^{-1} + \mathbf{\Phi}_\Omega^* \mathbf{\Phi}_\Psi (\mathbf{\Phi}_\Psi^* \mathbf{\Phi}_\Psi)^{-1} \mathbf{\Phi}_\Psi^* \mathbf{\Phi}_\Omega)$.

Note that $\mathbf{\Gamma}$ is a scalar since $|\Omega| = 1$, however, treating it as a matrix does not change our final result. Now from the reverse triangle inequality for any K-sparse signal $\mathbf{v}$, we have

$$||\mathbf{\Gamma}\mathbf{v}||_2 \geq ||\mathbf{I}\mathbf{v}||_2 - ||\mathbf{\Phi}_\Omega^* \mathbf{\Phi}_\Psi (\mathbf{\Phi}_\Psi^* \mathbf{\Phi}_\Psi)^{-1} \mathbf{\Phi}_\Psi^* \mathbf{\Phi}_\Omega \mathbf{v}||_2 \tag{11}$$

$$\geq \left(1 - \frac{\delta_K^2}{(1 - \delta_{K-1})}\right) ||\mathbf{v}||_2 \tag{12}$$

using RIP bounds to upper bound the right term and utilizing $\mathbf{\Phi}_\Omega^*\mathbf{\Phi}_\Omega = 1$. We also use Cauchy-Schwartz's inquality in a sequential cascaded manner. We use these techniques (sequentially applying Cauchy-Schwartz's followed by RIP) heavily from here on and in most of the bounds that we obtain in our proofs. Thus, we have

$$||\mathbf{\Gamma}^{-1}\mathbf{v}||_2 \leq \left(\frac{1-\delta_{K-1}}{1-\delta_{K-1}-\delta_K^2}\right)||\mathbf{v}||_2$$

Now from Equation. 10, we have

$$\begin{aligned}
&||\mathbf{\Pi}_\Omega^{-1}\mathbf{x}_{|\Psi}||_2 && (13)\\
&= ||((\mathbf{\Phi}_\Psi^*\mathbf{\Phi}_\Psi)^{-1} - (\mathbf{\Phi}_\Psi^*\mathbf{\Phi}_\Psi)^{-1}\mathbf{\Phi}_\Psi^*\mathbf{\Phi}_\Omega\mathbf{\Gamma}^{-1}\mathbf{\Phi}_\Omega^*\mathbf{\Phi}_\Psi(\mathbf{\Phi}_\Psi^*\mathbf{\Phi}_\Psi)^{-1})\mathbf{x}_{|\Psi}||_2 && (14)\\
&\leq ||((\mathbf{\Phi}_\Psi^*\mathbf{\Phi}_\Psi)^{-1}\mathbf{x}_{|\Psi}||_2 + ||(\mathbf{\Phi}_\Psi^*\mathbf{\Phi}_\Psi)^{-1}\mathbf{\Phi}_\Psi^*\mathbf{\Phi}_\Omega\mathbf{\Gamma}^{-1}\mathbf{\Phi}_\Omega^*\mathbf{\Phi}_\Psi(\mathbf{\Phi}_\Psi^*\mathbf{\Phi}_\Psi)^{-1}\mathbf{x}_{|\Psi}||_2 && (15)\\
&\leq \left(\frac{1}{1-\delta_{K-1}}\right)||\mathbf{x}_{|\Psi}||_2 + \left(\frac{\delta_K^2}{(1-\delta_{K-1}-\delta_K^2)(1-\delta_{K-1})}\right)||\mathbf{x}_{|\Psi}||_2 && (16)\\
&\leq \left(\frac{1}{1-\delta_{K-1}-\delta_K^2}\right)||\mathbf{x}_{|\Psi}||_2 && (17)
\end{aligned}$$

We first find an upper bound for $||\mathbf{Q}\mathbf{x}_{|\Psi}||_2$

$$\begin{aligned}
||\mathbf{Q}\mathbf{x}_{|\Psi}||_2 &= ||-(\mathbf{\Phi}_\Omega^*\mathbf{\Phi}_\Omega)^{-1}\mathbf{\Phi}_\Omega^*\mathbf{\Phi}_\Psi\mathbf{\Pi}_\Omega^{-1}\mathbf{x}_{|\Psi}||_2 && (18)\\
&\leq \left(\frac{\delta_K}{1-\delta_{K-1}-\delta_K^2}\right)||\mathbf{x}_{|\Psi}||_2 && (19)
\end{aligned}$$

We now bound $||\mathbf{P}\mathbf{x}_{|\Omega}||_2$ using RIP conditions and $\mathbf{\Phi}_\Omega^*\mathbf{\Phi}_\Omega = 1$.

$$\begin{aligned}
||\mathbf{P}\mathbf{x}_{|\Omega}||_2 &= \left((\mathbf{\Phi}_\Omega^*\mathbf{\Phi}_\Omega)^{-1} + (\mathbf{\Phi}_\Omega^*\mathbf{\Phi}_\Omega)^{-1}\mathbf{\Phi}_\Omega^*\mathbf{\Phi}_\Psi\mathbf{\Pi}_\Omega^{-1}\mathbf{\Phi}_\Psi^*\mathbf{\Phi}_\Omega\mathbf{\Phi}_\Omega^*\mathbf{\Phi}_\Omega\right)||\mathbf{x}_{|\Omega}||_2 && (20)\\
&\lesseqgtr \left(1 \pm \frac{\delta_K^3}{1-\delta_{K-1}-\delta_K^2}\right)||\mathbf{x}_{|\Omega}||_2 && (21)
\end{aligned}$$

From the previous two inequalities and Equation. 8 and also putting

$$\eta = \left(\frac{\delta_K}{1-\delta_{K-1}-\delta_K^2}\right)$$

, we have

$$||((\mathbf{\Phi}_F^*\mathbf{\Phi}_F)^{-1})^\Omega\mathbf{x}||_2 \lesseqgtr \left(1 \pm \delta_K^2\eta\right)||\mathbf{x}_{|\Omega}||_2 \pm \eta||\mathbf{x}_{|\Psi}||_2 \tag{22}$$

Q.E.D

2 Proof of Theorem 1

Proof. Given T, we define $F = (T \cap \lambda^*) \subseteq \lambda^*$ and $Z = T \backslash \lambda^*$ with $|F| = \gamma \leq K$ and $|Z| \leq |T| = z$, thus having $T = F \cup Z$. We also have $\Omega \subseteq F$ with $|\Omega| = 1$. Note that $F \cap Z = \lambda^* \cap Z = 0$. Also, we have $\mathbf{b}_T = (\mathbf{\Phi}_T^*\mathbf{\Phi}_T)^{-1}\mathbf{\Phi}_T^*\mathbf{\Phi}_{\lambda^*}\mathbf{x}$. Putting $\mathbf{\Phi}_T = [\mathbf{\Phi}_F\mathbf{\Phi}_Z]$

Our strategy is to find a condition such that the lower bound for $||\mathbf{b}_{|\Omega}||_2$ is greater than the upper bound for $||\mathbf{b}_{|\Psi}||_2$. This would force the true component Ω into the top K elements chosen during pruning.

We have,

$$\mathbf{b}_{|T} = \left(\begin{bmatrix} \mathbf{\Phi}_F^* \\ \mathbf{\Phi}_Z^* \end{bmatrix} [\mathbf{\Phi}_F \;\; \mathbf{\Phi}_Z] \right)^{-1} \begin{bmatrix} \mathbf{\Phi}_F^* \\ \mathbf{\Phi}_Z^* \end{bmatrix} \mathbf{u} \tag{23}$$

$$= \left(\begin{bmatrix} \mathbf{\Phi}_F^*\mathbf{\Phi}_F & \mathbf{\Phi}_F^*\mathbf{\Phi}_Z \\ \mathbf{\Phi}_Z^*\mathbf{\Phi}_F & \mathbf{\Phi}_Z^*\mathbf{\Phi}_Z \end{bmatrix} \right)^{-1} \begin{bmatrix} \mathbf{\Phi}_F^* \\ \mathbf{\Phi}_Z^* \end{bmatrix} \mathbf{u} \tag{24}$$

$$= \left(\begin{bmatrix} \mathbf{A}_{FF} & \mathbf{A}_{FZ} \\ \mathbf{A}_{ZF} & \mathbf{A}_{ZZ} \end{bmatrix} \right)^{-1} \begin{bmatrix} \mathbf{\Phi}_F^* \\ \mathbf{\Phi}_Z^* \end{bmatrix} \mathbf{u} \tag{25}$$

with $\mathbf{A}_{FF} = \mathbf{\Phi}_F^*\mathbf{\Phi}_F$, $\mathbf{A}_{FZ} = \mathbf{\Phi}_F^*\mathbf{\Phi}_Z$,$\mathbf{A}_{ZF} = \mathbf{\Phi}_Z^*\mathbf{\Phi}_F$ and $\mathbf{A}_{ZZ} = \mathbf{\Phi}_Z^*\mathbf{\Phi}_Z$.

$$\left(\begin{bmatrix} \mathbf{A}_{FF} & \mathbf{A}_{FZ} \\ \mathbf{A}_{ZF} & \mathbf{A}_{ZZ} \end{bmatrix} \right)^{-1} = \left(\begin{bmatrix} \mathbf{P} & \mathbf{Q} \\ \mathbf{R} & \mathbf{S} \end{bmatrix} \right) \tag{26}$$

where

$$\mathbf{P} = \mathbf{A}_{FF}^{-1} + \mathbf{A}_{FF}^{-1}\mathbf{A}_{FZ}\mathbf{\Sigma}^{-1}\mathbf{A}_{ZF}\mathbf{A}_{FF}^{-1} \tag{27}$$

$$\mathbf{Q} = -\mathbf{A}_{FF}^{-1}\mathbf{A}_{FZ}\mathbf{\Sigma^{-1}} \tag{28}$$

$$\mathbf{R} = -\mathbf{\Sigma}^{-1}\mathbf{A}_{ZF}\mathbf{A}_{FF}^{-1} \tag{29}$$

$$\mathbf{S} = \mathbf{\Sigma}^{-1} \tag{30}$$

$$\mathbf{\Sigma} = \mathbf{A}_{ZZ} - \mathbf{A}_{ZF}\mathbf{A}_{FF}^{-1}\mathbf{A}_{FZ} \tag{31}$$

using the block inversion formula

We therefore have,

$$\mathbf{b}_{|T} = \left(\begin{bmatrix} \mathbf{P} & \mathbf{Q} \\ \mathbf{R} & \mathbf{S} \end{bmatrix} \right) \begin{bmatrix} \mathbf{\Phi}_F^* \\ \mathbf{\Phi}_Z^* \end{bmatrix} \mathbf{\Phi}_{\lambda^*}\mathbf{x} \tag{32}$$

$$= \begin{pmatrix} \mathbf{P}\mathbf{\Phi}_F^*\mathbf{\Phi}_{\lambda^*} + \mathbf{Q}\mathbf{\Phi}_Z^*\mathbf{\Phi}_{\lambda^*} \\ \mathbf{R}\mathbf{\Phi}_F^*\mathbf{\Phi}_{\lambda^*} + \mathbf{S}\mathbf{\Phi}_Z^*\mathbf{\Phi}_{\lambda^*} \end{pmatrix} \mathbf{x} \tag{33}$$

We are interested in comparing $\mathbf{b}_{|\Omega} = (\mathbf{P}\mathbf{\Phi}_F^*\mathbf{\Phi}_{\lambda^*} + \mathbf{Q}\mathbf{\Phi}_Z^*\mathbf{\Phi}_{\lambda^*})^{\Omega}\mathbf{x}$ and $\mathbf{b}_{|Z} = (\mathbf{R}\mathbf{\Phi}_F^*\mathbf{\Phi}_{\lambda^*} + \mathbf{S}\mathbf{\Phi}_Z^*\mathbf{\Phi}_{\lambda^*})\mathbf{x}$.

Before moving on, we digress briefly to obtain an upper bound on $||\mathbf{\Sigma}^{-1}\mathbf{v}||_2$, where $\mathbf{v}$ is K-sparse.

$$\mathbf{\Sigma} = \mathbf{A}_{ZZ} + \mathbf{A}_{ZF}(-\mathbf{A}_{FF}^{-1})\mathbf{A}_{FZ}$$

Using Woodbury's matrix inversion lemma, we have

$$\mathbf{\Sigma}^{-1} = (\mathbf{A}_{ZZ}^{-1} - \mathbf{A}_{ZZ}^{-1}\mathbf{A}_{ZF}\mathbf{C}^{-1}\mathbf{A}_{FZ}\mathbf{A}_{ZZ}^{-1})$$

where, $\mathbf{C} = -\mathbf{A}_{FF}^{-1} + \mathbf{A}_{FZ}\mathbf{A}_{ZZ}^{-1}\mathbf{A}_{ZF}$,

$$||\mathbf{C}\mathbf{v}||_2 \geq ||\mathbf{A}_{FF}^{-1}\mathbf{v}||_2 - ||\mathbf{A}_{FZ}\mathbf{A}_{ZZ}^{-1}\mathbf{A}_{ZF}\mathbf{v}||_2 \tag{34}$$

we see that

$$||\mathbf{C}\mathbf{v}||_2 \geq \left(\frac{1}{(1+\delta_K)} - \frac{\delta_{Z+K}^2}{(1-\delta_Z)} \right) ||\mathbf{v}||_2$$

therefore,

$$||\mathbf{C}^{-1}\mathbf{v}||_2 \leq \left(\frac{(1+\delta_K)(1-\delta_Z)}{(1-\delta_Z) - (1+\delta_K)\delta_{Z+K}^2} \right) ||\mathbf{v}||_2$$

Further, setting $\mathbf{D} = \mathbf{A}_{ZZ}^{-1}\mathbf{A}_{ZF}\mathbf{C}^{-1}\mathbf{A}_{FZ}\mathbf{A}_{ZZ}^{-1}$ so that $\Sigma^{-1} = (\mathbf{A}_{ZZ}^{-1} - \mathbf{D})$. Using RIP conditions we obtain

$$||\mathbf{Dv}||_2 \leq \left(\frac{(1+\delta_K)(1-\delta_Z)\delta_{Z+K}^2}{[(1-\delta_Z)-(1+\delta_K)\delta_{Z+K}^2](1-\delta_Z)^2} \right) ||\mathbf{v}||_2 \tag{35}$$

$$= \left(\frac{(1+\delta_K)\delta_{Z+K}^2}{[(1-\delta_Z)-(1+\delta_K)\delta_{Z+K}^2](1-\delta_Z)} \right) ||\mathbf{v}||_2 \tag{36}$$

Thus,

$$||\mathbf{\Sigma}^{-1}\mathbf{v}||_2 = ||\mathbf{A}_{ZZ}^{-1}\mathbf{v} - \mathbf{Dv}||_2 \tag{37}$$

$$\leq ||\mathbf{A}_{ZZ}^{-1}\mathbf{v}||_2 + ||\mathbf{Dv}||_2 \tag{38}$$

$$\leq \kappa||\mathbf{v}||_2 \tag{39}$$

where $\kappa = \left(\frac{1}{(1-\delta_Z)} + \frac{(1+\delta_K)\delta_{Z+K}^2}{[(1-\delta_Z)-(1+\delta_K)\delta_{Z+K}^2](1-\delta_Z)} \right)$

We now move on to upper bound $||\mathbf{b}_{|Z}||_2$.

$$||\mathbf{b}_{|Z}||_2 = ||(\mathbf{R\Phi}_F^*\mathbf{\Phi}_{\lambda^*} + \mathbf{S\Phi}_Z^*\mathbf{\Phi}_{\lambda^*})\mathbf{x}||_2 \tag{40}$$

$$\leq ||\mathbf{R\Phi}_F^*\mathbf{\Phi}_{\lambda^*}\mathbf{x}||_2 + ||\mathbf{S\Phi}_Z^*\mathbf{\Phi}_{\lambda^*}\mathbf{x}||_2 \tag{41}$$

One can use the restricted isometric property to bound $||\mathbf{\Phi}_F^*\mathbf{\Phi}_{\lambda^*}\mathbf{x}||_2$ as follows.

$$(1-2\delta_K)||\mathbf{x}||_2 \leq ||\mathbf{\Phi}_F^*\mathbf{\Phi}_{\lambda^*}\mathbf{x}||_2 \leq (1+2\delta_K)||\mathbf{x}||_2 \tag{42}$$

We upper bound both right hand terms and use inequality 42. Using Equation. 29, we obtain

$$||\mathbf{R\Phi}_F^*\mathbf{\Phi}_{\lambda^*}\mathbf{x}||_2 = ||-\Sigma^{-1}\mathbf{A}_{ZF}\mathbf{A}_{FF}^{-1}\mathbf{\Phi}_F^*\mathbf{\Phi}_{\lambda^*}\mathbf{x}||_2 \tag{43}$$

$$\leq \kappa\frac{\delta_{z+K}(1+2\delta_K)}{(1-\delta_K)}||\mathbf{x}||_2 \tag{44}$$

Using Proposition 3.1 and Proposition 3.2 (Approximate Orthogonality) from Nedell and Tropp [1], we obtain a lower bound on the first term and upper bounds on the second and third terms. Recall that $F \subseteq \lambda^*$, thus $|F| \leq |\lambda^*| = K$.

Using Equation. 31, we find

$$||\mathbf{S\Phi}_Z^*\mathbf{\Phi}_{\lambda^*}\mathbf{x}||_2 = ||\mathbf{\Sigma}^{-1}\mathbf{\Phi}_Z^*\mathbf{\Phi}_{\lambda^*}\mathbf{x}||_2 \tag{45}$$

$$\leq \kappa\delta_{Z+K}||\mathbf{x}||_2 \tag{46}$$

Combining the previous two inequalities , we have

$$||\mathbf{b}_{|Z}||_2 \tag{47}$$

$$= ||(\mathbf{R\Phi}_F^*\mathbf{\Phi}_{\lambda^*} + \mathbf{S\Phi}_Z^*\mathbf{\Phi}_{\lambda^*})\mathbf{x}||_2 \tag{48}$$

$$\leq \kappa\left(\frac{\delta_{Z+K}(1+2\delta_K)}{(1-\delta_K)} + \delta_{Z+K} \right)||\mathbf{x}||_2 \tag{49}$$

We now find a lower bound for $||\mathbf{b}_{|\Omega}||_2$.

$$||\mathbf{b}_{|\Omega}||_2 = ||(\mathbf{P})^{\Omega}\mathbf{\Phi}_F^*\mathbf{\Phi}_{\lambda^*} + (\mathbf{Q})^{\Omega}\mathbf{\Phi}_Z^*\mathbf{\Phi}_{\lambda^*}\mathbf{x}||_2 \tag{50}$$

$$\geq ||(\mathbf{P})^{\Omega}\mathbf{\Phi}_F^*\mathbf{\Phi}_{\lambda^*}\mathbf{x}||_2 - ||(\mathbf{Q})^{\Omega}\mathbf{\Phi}_Z^*\mathbf{\Phi}_{\lambda^*}\mathbf{x}||_2 \tag{51}$$

Using Equation. 27 and the reverse triangle inequality, we obtain a lower bound on the first term

$$||(\mathbf{P})^{\Omega}\mathbf{\Phi}_F^*\mathbf{\Phi}_{\lambda^*}\mathbf{x}||_2 \tag{52}$$
$$\geq ||(\mathbf{A}_{FF}^{-1})^{\Omega}\mathbf{\Phi}_F^*\mathbf{\Phi}_{\lambda^*}\mathbf{x}||_2 - ||(\mathbf{A}_{FF}^{-1})^{\Omega}\mathbf{A}_{FZ}\mathbf{\Sigma}^{-1}\mathbf{A}_{ZF}\mathbf{A}_{FF}^{-1}\mathbf{\Phi}_F^*\mathbf{\Phi}_{\lambda^*}\mathbf{x}||_2 \tag{53}$$
$$\geq (1-2\delta_K)\left[(1-\delta_K^2\eta)||\mathbf{x}_{|\Omega}||_2 - \eta||\mathbf{x}_{|\Psi}||_2\right] \tag{54}$$
$$-\left(\frac{\delta_{Z+K}^2(1+2\delta_K)}{(1-\delta_K)}\right)\kappa\left[(1+\delta_K^2\eta)||\mathbf{x}_{|\Omega}||_2 + \eta||\mathbf{x}_{|\Psi}||_2\right] \tag{55}$$

Here we use Lemma 1, along with Proposition 3.2 from Needel and Tropp [1].

Using Equation. 28 and the triangle inequality, we obtain an upper bound on the second term

$$||(\mathbf{Q})^{\Omega}\mathbf{\Phi}_Z^*\mathbf{\Phi}_{\lambda^*}\mathbf{x}||_2 \tag{56}$$
$$= ||(-\mathbf{A}_{FF}^{-1})^{\Omega}\mathbf{A}_{FZ}\mathbf{\Sigma^{-1}}\mathbf{\Phi}_Z^*\mathbf{\Phi}_{\lambda^*})\mathbf{x}||_2 \tag{57}$$
$$\leq \delta_K\delta_{Z+K}\kappa\left[(1+\delta_K^2\eta)||\mathbf{x}_{|\Omega}||_2 + \eta||\mathbf{x}_{|\Psi}||_2\right] \tag{58}$$

Here we use Lemma 1, Cauchy's inequality and Proposition 3.2 from Needel and Tropp [1]. Combining the last two inequalities and Equation. 51 we summarize,

$$||\mathbf{b}_{|\Omega}||_2 \tag{59}$$
$$= ||(\mathbf{P})^{\Omega}\mathbf{\Phi}_F^*\mathbf{\Phi}_{\lambda^*} + (\mathbf{Q})^{\Omega}\mathbf{\Phi}_Z^*\mathbf{\Phi}_{\lambda^*}\mathbf{x}||_2 \tag{60}$$
$$\geq (1-2\delta_K)\left[(1-\delta_K^2\eta)||\mathbf{x}_{|\Omega}||_2 - \eta||\mathbf{x}_{|\Psi}||_2\right] \tag{61}$$
$$-\left(\frac{\delta_{Z+K}^2(1+2\delta_K)}{(1-\delta_K)}\right)\kappa\left[(1+\delta_K^2\eta)||\mathbf{x}_{|\Omega}||_2 + \eta||\mathbf{x}_{|\Psi}||_2\right] \tag{62}$$
$$-\delta_K\delta_{Z+K}\kappa\left[(1+\delta_K^2\eta)||\mathbf{x}_{|\Omega}||_2 + \eta||\mathbf{x}_{|\Psi}||_2\right] \tag{63}$$

We now compare the lower bound on $||\mathbf{b}_{|\Omega}||_2$ to the upper bound on $||\mathbf{b}_{|Z}||_2$, and also use the triangle inequality on $||\mathbf{x}||_\mathbf{2}$. We arrive at the inequality

$$(1-2\delta_K)\left[(1-\delta_K^2\eta)||\mathbf{x}_{|\Omega}||_2 - \eta||\mathbf{x}_{|\Psi}||_2\right] \tag{64}$$
$$-\left(\frac{\delta_{Z+K}^2(1+2\delta_K)}{(1-\delta_K)}\right)\kappa\left[(1+\delta_K^2\eta)||\mathbf{x}_{|\Omega}||_2 + \eta||\mathbf{x}_{|\Psi}||_2\right] \tag{65}$$
$$-\delta_K\delta_{Z+K}\kappa\left[(1+\delta_K^2\eta)||\mathbf{x}_{|\Omega}||_2 + \eta||\mathbf{x}_{|\Psi}||_2\right] \tag{66}$$
$$\geq \left[\kappa\frac{\delta_{Z+K}(1+2\delta_K)}{(1-\delta_K)} + \kappa\delta_{Z+K}\right](||\mathbf{x}_{|\Omega}||_2 + ||\mathbf{x}_{|\Psi}||_2) \tag{67}$$

Rearranging, we get

$$\left((1-2\delta_K)(1-\delta_K^2\eta) - \left(\frac{\delta_{Z+K}^2(1+2\delta_K)}{(1-\delta_K)}\right)\kappa(1+\delta_K^2\eta)\right)||\mathbf{x}_{|\Omega}||_2 \tag{68}$$
$$+\left(-\delta_K\delta_{Z+K}\kappa(1+\delta_K^2\eta) - \left[\kappa\frac{\delta_{Z+K}(1+2\delta_K)}{(1-\delta_K)} + \kappa\delta_{Z+K}\right]\right)||\mathbf{x}_{|\Omega}||_2 \tag{69}$$
$$\geq \left((1-2\delta_K)\eta + \left(\frac{\delta_{Z+K}^2(1+2\delta_K)}{(1-\delta_K)}\right)\kappa\eta\right)||\mathbf{x}_{|\Psi}||_2 \tag{70}$$
$$+\left(\delta_K\delta_{Z+K}\kappa\eta + \left[\kappa\frac{\delta_{Z+K}(1+2\delta_K)}{(1-\delta_K)} + \kappa\delta_{Z+K}\right]\right)||\mathbf{x}_{|\Psi}||_2 \tag{71}$$

Combining all scalars into β, we arrive at

$$\frac{|\mathbf{x}_{|\Omega}|}{||\mathbf{x}_{|\Psi}||_2} \geq \beta$$

,since $|\Omega| = 1$. We find that $\beta = 0$ for $\{\delta_K, \delta_Z\} = \delta_{Z+K} = 0$ and $\beta = 0.68$ for $\{\delta_K, \delta_Z\} \leq \delta_{Z+K} \leq 0.1$. Thus, for $\delta_{Z+K} = 0$, we would have

$$||\mathbf{b}_{|\Omega}||_2 \geq ||\mathbf{b}_{|Z}||_2 \tag{72}$$

Note that this holds for all $\forall F \subseteq \lambda^*$ and $\forall \Omega \subseteq F$ with $|\Omega| = 1$. Now, $|F| \leq K$, thus the set of top K elements of $\mathbf{b}$ would always contain F. Thus,

$$F \subseteq supp(\mathbf{b}_K) = \lambda$$

Thus trivially, in case of a perfect isometry, the top K elements would always contain F. However, for $\{\delta_K, \delta_Z\} \leq \delta_{Z+K} \leq 0.03$, we find $\beta \approxeq 0.1$. Hence, signal components satisfying Equation. 72 would be included in the top K components picked during pruning in a particular iteration. In particular, $\forall \Omega \in F \subseteq \lambda^*$, if Equation. 72 holds then,

$$\Omega \in \lambda$$

Q.E.D

3 Proof of Theorem 2

Proof. We have $\Upsilon^i = \lambda^i \cap \lambda^*$ and $\Omega \notin \lambda^i$. Now from Theorem 2.4, $\Omega \in \lambda^{i+1}$ *i.e.* Ω is contained in the new support estimate. Therefore, we have

$$|\Upsilon^{i+1}| = |\lambda^{i+1} \cap \lambda^*| \tag{73}$$

$$\geq |(\lambda^i \cap \lambda^*) \cup \Omega| \geq |\lambda^i \cap \lambda^*| = |\Upsilon^i| \tag{74}$$

The second inequality holds since there may exist multiple Ω s.t. $\mathbf{x}_{|\Omega}$ is β-strong w.r.t some Ψ. Q.E.D

References

[1] D. Needell and J. A. Tropp, "Cosamp: iterative signal recovery from incomplete and inaccurate samples," *Commun. ACM*, vol. 53, no. 12, pp. 93100, Dec. 2010.

[2] D. Vats and R. Baraniuk, When in doubt, swap: High-dimensional sparse recovery from correlated measurements, in Proceedings of the Advances in Neural Information Processing Systems (NIPS), 2013.